Kuwait University	Math. 101	October 31, 2011
Dept. of Mathematics	First Midterm Exam	Duration 90 Minutes

Calculators and mobile phones are not allowed

Answer the following questions.

1. (2 *Points each*) Find the limit, if it exists.

$$(a)\ \lim_{x\to 2}\frac{\sqrt{2x+5}-2}{x^2-9} \qquad (b)\ \lim_{x\to 0}\frac{3x+\tan 2x}{2x+3\sin x}.$$

2. (4 *Points*) Use the Intermediate Value Theorem to show that the graphs of the functions

$$f(x)=2+3x-\pi\cos x \quad \text{and} \quad g(x)=2x+\sin x$$

intersect.

3. (4 *Points*) Let $f(x)=\sqrt{x+1}$. Use the definition of derivative to find $f'(3)$.

4. (4 *Points*) Let $f(x)=x^3\tan^2 x+\sqrt{\sec(2x)}$. Find $f'(x)$.

5. (4 *Points*) Let $f(x)=x^{7/3}-7x^{1/3}$. Find all the points on the graph of f at which

   (a) the tangent line is horizontal

   (b) the tangent line is vertical.

6. (5 *Points*) Find the x-coordinate of the points at which the function f is discontinuous, where

$$f(x)=\begin{cases}\dfrac{2x+1}{x+3}, & \text{if } x<0,\\[2ex] \dfrac{x^2-1}{x^2+x-2}, & \text{if } x\ge 0.\end{cases}$$

Classify the types of discontinuity of f as removable, jump or infinite.

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1.a $\lim_{x\to 2} \dfrac{\sqrt{2x+5}-2}{x^2-9} = \dfrac{\lim_{x\to 2}\left(\sqrt{2x+5}-2\right)}{\lim_{x\to 2}\left(x^2-9\right)} = \dfrac{3-2}{4-9} = \boxed{-\dfrac{1}{5}}.$

1.b $\lim_{x\to 0} \dfrac{3x+\tan 2x}{2x+3\sin x} = \lim_{x\to 0} \dfrac{x\left(3+\dfrac{2\tan 2x}{2x}\right)}{x\left(2+3\dfrac{\sin x}{x}\right)} = \lim_{x\to 0} \dfrac{3+2\left(\dfrac{\tan 2x}{2x}\right)}{2+3\left(\dfrac{\sin x}{x}\right)} = \dfrac{3+2}{2+3} = \boxed{1.}$

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2. Let $h(x) = f(x) - g(x) = 2 + x - \pi\cos x - \sin x$.

h is continuous on $\mathbb{R}$ therefore it is continuous on $\left[0, \frac{\pi}{2}\right]$. Moreover,

$h(0)\, h\left(\frac{\pi}{2}\right) = (2-\pi)\left(1+\frac{\pi}{2}\right) < 0,$

$\overset{IVT}{\Longrightarrow}$ there exist a number c in $\left(0, \frac{\pi}{2}\right)$ such that $h(c) = 0$ or $f(c) = g(c)$.

3.

$$\lim_{x\to 3} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to 3} \frac{x+1-4}{(x-3)\left(\sqrt{x+1}+2\right)}$$

$$= \lim_{x\to 3} \frac{1}{\left(\sqrt{x+1}+2\right)} = \frac{1}{4} \Longrightarrow f'(3) = \boxed{\frac{1}{4}.}$$

4.

$$f'(x) = 3x^2\tan^2 x + x^3\left(2\tan x\sec^2 x\right) + \frac{2\sec 2x\tan 2x}{2\sqrt{\sec 2x}}$$

$$= \boxed{3x^2\tan^2 x + 2x^3\tan x\sec^2 x + \sqrt{\sec 2x}\tan 2x.}$$

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5. $f(x) = x^{7/3} - 7x^{1/3}$.

The domain of f is $\mathbb{R}$ and $\boxed{f'(x) = \frac{7}{3}x^{4/3} - \frac{7}{3}x^{-2/3} = \dfrac{7\left(x^2-1\right)}{3x^{2/3}}.}$

a. $f'(x) = 0$ when $x = \pm 1$. The graph of f has HTL at the points $\boxed{(\pm 1, \mp 6).}$

b. f is continuous at $x = 0$ and $\lim_{x\to 0}|f'(x)| = \infty$. The graph of f has VTL at the point $\boxed{(0,0)}$

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6. The function is discontinuous at $x = -3$, $x = 0$, and $x = 1$.

$\lim_{x\to -3^{\pm}} f(x) = \mp\infty.$ $\boxed{\text{The function has an infinite discontinuity at } x = -3.}$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} \dfrac{2x+1}{x+3} = \boxed{\dfrac{1}{3}}$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0} \dfrac{x^2-1}{x^2+x-2} = \boxed{\dfrac{1}{2}.}$

$\boxed{\text{The function has a jump discontinuity at } x = 0.}$

$f(1)$ does not exist and

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x^2+x-2} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x+2)} = \lim_{x\to 1} \frac{x+1}{x+2} = \boxed{\frac{2}{3}}.$$

$\boxed{\text{The function has a removable discontinuity at } x = 1.}$